SUFFICIENT CONDITIONS FOR LABELLED 0–1 LAWS

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ABSTRACT. If $\mathbf{F}(x) = e^{\mathbf{G}(x)}$, where $\mathbf{F}(x) = \sum f(n)x^n$ and $\mathbf{G}(x) = \sum g(n)x^n$, with $0 \le g(n) = \mathrm{O}\big(n^{\theta n}/n!\big)$, $\theta \in (0,1)$, and $\gcd\big(n : g(n) > 0\big) = 1$, then $f(n) = \mathrm{o}(f(n-1))$.

This gives an answer to Compton's request in Question 8.3 [3] for an "easily verifiable sufficient condition" to show that an adequate class of structures has a labelled first-order 0–1 law, namely it suffices to show that the labelled component count function is $\mathrm{O}\big(n^{\theta n}\big)$ for some $\theta \in (0,1)$. It also provides the means to recursively construct an adequate class of structures with a labelled 0–1 law but not an unlabelled 0–1 law, answering Compton's Question 8.4.

1. Introduction

Exponentiating a power series can have the effect of smoothing out the behavior of the coefficients. In this paper we look at conditions on the growth of the coefficients of $\mathbf{G}(x) = \sum g(n)x^n$, where $g(n) \ge 0$, which ensure that $f(n-1)/f(n) \to \infty$, where $\mathbf{F}(x) = e^{\mathbf{G}(x)}$.

Useful notation will be $f(n) \prec g(n)$ for $f(n)$ eventually less than $g(n)$ and $f(n) \in \mathsf{RT}_\infty$ for $f(n-1)/f(n) \to \infty$; the notation RT stands for the ratio test.

2. The Coefficients of e^{poly}

Proposition 1. *Given*

$$\begin{aligned} \mathbf{G}(x) &:= g(1)x + \cdots + g(d)x^d, \quad g(i) \ge 0,\ g(d) > 0, \\ &\quad\ \textit{with } \gcd\big(j \le d : g(j) > 0\big) \ = \ 1 \\ \mathbf{F}(x) &:= \sum_{n \ge 0} f(n)x^n \ = \ e^{\mathbf{G}(x)}, \end{aligned}$$

the function $\mathbf{F}(x)$ is Hayman-admissible. Thus

$$f(n) \ \sim \ \frac{\mathbf{F}(r_n)}{{r_n}^n \cdot \sqrt{2\pi \mathbf{B}(r_n)}} \tag{1}$$

where r_n is the unique positive solution to

$$x \cdot \mathbf{G}'(x) \ = \ n,$$

and $\mathbf{B}(x) := x^2\mathbf{G}''(x) + x\mathbf{G}'(x)$.

Proof. Theorem X of Hayman [5] shows that $\mathbf{F}(x)$ is Hayman-admissible. Then the rest of the claim is an immediate consequence of Corollary II of [5] where the saddle-point method is applied to find the asymptotics of the coefficients of an admissible function. □

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Corollary 2. *For* $\mathbf{F}(x), \mathbf{G}(x)$ *as in the above proposition,*

(a) $f(n) \in \mathsf{RT}_\infty$,

(b) $f(n) \;=\; \exp\Big(-\dfrac{n\log n}{d}\big(1+\mathrm{o}(1)\big)\Big)$.

Proof. Item (a) follows immediately from Corollary IV of Hayman [5].

For item (b) one uses $r_n\mathbf{G}'(r_n) \;=\; n$ to obtain:

$$
\begin{aligned}
\Big(\frac{n}{cdg(d)}\Big)^{1/d} \;&\preceq\; r_n \;\le \Big(\frac{n}{dg(d)}\Big)^{1/d} \quad \text{for } c>1\\
r_n \;&=\; \big(1+\mathrm{o}(1)\big)\Big(\frac{n}{dg(d)}\Big)^{1/d}\\
{r_n}^n \;&=\; \big(1+\mathrm{o}(1)\big)^n\Big(\frac{n}{dg(d)}\Big)^{n/d}\\
\mathbf{B}(r_n) \;&=\; \big(1+\mathrm{o}(1)\big)d^2g(d)\Big(\frac{n}{dg(d)}\Big) \;=\; \big(1+\mathrm{o}(1)\big)dn\\
\mathbf{G}(r_n) \;&=\; \big(1+\mathrm{o}(1)\big)g(d){r_n}^d \;=\; \big(1+\mathrm{o}(1)\big)\frac{n}{d}\\
\mathbf{F}(r_n) \;&=\; \exp\Big(\frac{n}{d}\big(1+\mathrm{o}(1)\big)\Big).
\end{aligned}
$$

Apply these results to (1).

□

3. Some Technical Lemmas

Now we drop the assumption that $\mathbf{G}(x)$ is a polynomial, but keep the requirement

$$\gcd\big(n : g(n) > 0\big) \;=\; 1. \tag{2}$$

This implies that $f(n) \succ 0$.

Choose a positive integer $L \ge 2$ sufficiently large so

$$n > L \;\Rightarrow\; [x^n]\exp\Big(g(1)x + \cdots + g(L)x^L\Big) \;>\; 0. \tag{3}$$

Given $\ell > L$ with $g(\ell) > 0$ let

$$
\begin{aligned}
\mathbf{G}_0(x) \;&:=\; \sum_{n\ge 1} g_0(n)x^n \;:=\; \sum_{1\le n\le \ell} g(n)x^n\\
\mathbf{F}_0(x) \;&:=\; \sum_{n\ge 0} f_0(n)x^n \;:=\; \exp(\mathbf{G}_0(x))\\
\mathbf{G}_1(x) \;&:=\; \sum_{n\ge 1} g_1(n)x^n \;:=\; \sum_{n\ge \ell+1} g(n)x^n\\
\mathbf{F}_1(x) \;&:=\; \sum_{n\ge 0} f_1(n)x^n \;:=\; \exp(\mathbf{G}_1(x)).
\end{aligned} \tag{4}
$$

Lemma 3. *Suppose* $r \ge -1$ *is such that*

$$ng(n) \;=\; \mathrm{O}\big(f_0(n+r)\big). \tag{5}$$

Then

$$nf_1(n) \;=\; \mathrm{O}\big(f(n+r)\big).$$

Proof. In view of (3) and (5) we can choose C_r such that

$$ng(n) \;\leq\; C_r f_0(n+r) \quad \text{for } n+r \geq L+1. \tag{6}$$

Differentiating (4) gives

$$\begin{aligned}
nf_1(n) &= \sum_{j=\ell+1}^{n} jg(j) \cdot f_1(n-j) \\
&\leq C_r \sum_{j=\ell+1}^{n} f_0(j+r) \cdot f_1(n-j) \quad \text{by (6)} \\
&\leq C_r \sum_{j=0}^{n+r} f_0(j) \cdot f_1(n+r-j) \\
&= C_r f(n+r),
\end{aligned}$$

the last line following from $\mathbf{F}(x) = \mathbf{F}_0(x) \cdot \mathbf{F}_1(x)$. □

Lemma 4. *Suppose for every integer $r \geq -1$*

$$ng(n) \;=\; \mathrm{O}\big(f_0(n+r)\big).$$

Then $f(n-1)/f(n) \to \infty$.

Proof. Since $f_0(n) \in \mathsf{RT}_\infty$ by Corollary 2 there is a monotone decreasing function $\varepsilon(n)$ such that for any sufficiently large M we have $\varepsilon(n) > f_0(n)/f_0(n-1)$ for $n \geq M$, and $\varepsilon(n) \to 0$ as $n \to \infty$.

Thus

$$\begin{aligned}
f(n) &= \sum_{0\leq j\leq n} f_0(j)f_1(n-j) \\
&= \sum_{0\leq j\leq M-1} f_0(j)f_1(n-j) \;+\; \sum_{M\leq j\leq n} f_0(j)f_1(n-j) \\
&\leq \mathrm{o}\big(f(n-1)\big) \;+\; \varepsilon(M) \sum_{M\leq j\leq n} f_0(j-1)f_1(n-j) \\
&\qquad \text{by Lemma 3 and the choice of } \varepsilon \\
&\leq \mathrm{o}\big(f(n-1)\big) \;+\; \varepsilon(M)f(n-1).
\end{aligned}$$

Thus

$$\limsup_{n\to\infty} \frac{f(n)}{f(n-1)} \;\leq\; \varepsilon(M),$$

and as M can be arbitrarily large it follows that

$$\lim_{n\to\infty} \frac{f(n)}{f(n-1)} \;=\; 0.$$

□

4. Main Result

We are now in a position to prove the main result, making use of

$$n! \;=\; \exp\big(n \log n \cdot (1 + \mathrm{o}(1))\big),$$

which follows from Stirling's result.

Theorem 5. *Suppose* $\mathbf{F}(x) = \exp(\mathbf{G}(x))$ *with* $\mathbf{F}(x) = \sum_{n\geq 0} f(n)x^n$, $\mathbf{G}(x) = \sum_{n\geq 1} g(n)x^n$, *and* $f(n), g(n) \geq 0$. *Suppose also that* $\gcd\big(n : g(n) > 0\big) = 1$ *and that for some* $\theta \in (0,1)$

$$g(n) = \mathrm{O}\big(n^{\theta n}/n!\big).$$

Then

$$f(n) \in \mathsf{RT}_\infty.$$

Proof. From Corollary 2, for any integer $r \geq -1$ and any $\theta \in (0,1)$, by choosing $\ell > L$ such that $1/\ell < 1 - \theta$, we have

$$\begin{aligned} f_0(n+r) &= \exp\Big(-\frac{(n+r)\log(n+r)}{\ell}\big(1+\mathrm{o}(1)\big)\Big) \\ &= \exp\Big(-\frac{n\log n}{\ell}\big(1+\mathrm{o}(1)\big)\Big) \\ &\succ \frac{n^{\theta n}}{(n-1)!}\ . \end{aligned}$$

Thus $ng(n) = \mathrm{O}\big(f_0(n+r)\big)$. The Theorem then follows from Lemma 4. $\square$

5. Best Possible Result

The main result is in a natural sense the best possible.

Proposition 6. *Suppose* $t(n) \geq 0$ *with* $\gcd\big(n : t(n) > 0\big) = 1$ *is such that for any* $\theta \in (0,1)$

$$t(n) \ \neq\ \mathrm{O}\big(n^{\theta n}/n!\big).$$

Then there is a sequence $g(n) \geq 0$ *with* $\gcd\big(n : g(n) > 0\big) = 1$ *and* $g(n) \leq t(n)$ *but* $f(n) \notin \mathsf{RT}_\infty$, *where one has* $\mathbf{F}(x) = \exp(\mathbf{G}(x))$.

Proof. For $\theta \in (0,1)$ let

$$S(\theta) \ =\ \big\{n \geq 1 : t(n) > n^{\theta n}/n!\big\}\,.$$

Then $S(\theta)$ is an infinite set.

Let M be such that $\gcd\big(n \leq M : t(n) > 0\big) = 1$, and let

$$\begin{aligned} g_1(n) &:= \begin{cases} t(n) & \text{if } n \leq M \\ 0 & \text{if } n > M \end{cases} \\ \mathbf{G}_1(x) &:= \sum g_1(n)x^n \\ d_1 &:= \deg(\mathbf{G}_1(x)) \\ \mathbf{F}_1(x) &:= e^{\mathbf{G}_1(x)}. \end{aligned}$$

For $m \geq 2$ we give a recursive procedure to define polynomials $\mathbf{G}_m(x)$; then letting

$$\begin{aligned} d_m &:= \deg(\mathbf{G}_m(x)) \\ \mathbf{F}_m(x) &:= e^{\mathbf{G}_m(x)}, \end{aligned}$$

by Proposition 1

$$f_m(n) \ =\ \exp\Big(-\frac{n\log n}{d_m}\big(1+\mathrm{o}(1)\big)\Big).$$

To define $\mathbf{G}_{m+1}(x)$, having defined $\mathbf{G}_m(x)$, let

$$h_m(n) \ := \ \frac{1}{n!} \cdot n^{(1-1/2d_m)n}.$$

Then

$$\frac{h_m(n)}{f_m(n-1)} \ \rightarrow \ \infty \quad \text{as } n \rightarrow \infty.$$

Thus we can choose an integer $d_{m+1} > d_m$ such that

$$\begin{aligned} d_{m+1} \quad &\in \quad S\Big(1 - \frac{1}{2d_m}\Big) \\ h_m(d_{m+1}) \quad &> \quad f_m(d_{m+1} - 1). \end{aligned}$$

This ensures that $h_m(d_{m+1}) \le t(d_{m+1})$. Let

$$\mathbf{G}_{m+1} \ := \ \mathbf{G}_m(x) + h_m(d_{m+1})x^{d_{m+1}}.$$

Then

$$\frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} \ \ge \ \frac{h_m(d_{m+1})}{f_m(d_{m+1}-1)} \ > \ 1.$$

Now let $\mathbf{G}(x)$ be the nonnegative power series defined by the sequence of polynomials $\mathbf{G}_m(x)$; and let $\mathbf{F}(x) = e^{\mathbf{G}(x)}$. Then $g(n) \le t(n)$ but $f(n) \notin \mathsf{RT}_\infty$ as

$$\frac{f(d_{m+1})}{f(d_{m+1}-1)} \ = \ \frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} \ > \ 1.$$

□

6. Application to 0–1 laws

A class $\mathcal{K}$ of finite relational structures is *adequate* if it is closed under disjoint union and the extraction of components. One can view the structures as being *unlabelled* with the component count function $p_U(n)$ and the total count function $a_U(n)$, both counting up to isomorphism. The corresponding *ordinary* generating series are

$$\mathbf{P}_U(x) \ := \ \sum_{n \ge 1} p_U(n)x^n, \qquad \mathbf{A}_U(x) \ := \ \sum_{n \ge 0} a_U(n)x^n$$

connected by the fundamental equation

$$\mathbf{A}_U(x) \ = \ \prod_{j \ge 1} \big(1 - x^j\big)^{-p_U(j)}. \tag{7}$$

One can also view the structures as being *labelled* (in all possible ways) with the count functions $p_L(n)$ for the connected members of $\mathcal{K}$, and $a_L(n)$ for all members of $\mathcal{K}$. The corresponding *exponential* generating series are

$$\mathbf{P}_L(x) \ := \ \sum_{n \ge 1} p_L(n)x^n/n!, \qquad \mathbf{A}_L(x) \ := \ \sum_{n \ge 0} a_L(n)x^n/n!$$

connected by the fundamental equation

$$\mathbf{A}_L(x) \ = \ e^{\mathbf{P}_L(x)}. \tag{8}$$

All references to Compton in this section are to the two papers [3] and [4].

6.1. **Unlabelled 0–1 Laws for Adequate Classes.** Let $\mathcal{K}$ be an adequate class with unlabelled count functions and ordinary generating functions as described above. Compton showed that if the radius of convergence ρ_U of $\mathbf{A}_U(x)$ is positive then $\mathcal{K}$ has an unlabelled 0–1 law[1] iff $a_U(n) \in \mathsf{RT}_1$, that is,

$$\frac{a_U(n-1)}{a_U(n)} \ \to\ 1 \quad \text{as } n \to \infty.$$

$\mathcal{K}$ is finitely generated if $r = \sum p_U(n) < \infty$, that is, there are only finitely many connected structures in $\mathcal{K}$. In the finitely generated case the asymptotics for the coefficients $a_U(n)$ have long been known to have the simple polynomial form[2]

$$a_U(n) \ \sim\ Cn^{r-1} \tag{9}$$

provided $\gcd\big(n : p_U(n) > 0\big) = 1$. Item (9) leads to the fact that $a_U(n) \in \mathsf{RT}_1$, and hence to an unlabelled 0–1 law. In addition to using this result, Compton notes that the work of Bateman and Erdös [1] shows that if $p_U(n) \in \{0,1\}$, for all n, then one has $a_U(n) \in \mathsf{RT}_1$.

Both of these results were subsumed in the powerful result of Bell [2] which says that if $p_U(n)$ is polynomially bounded, that is, there is a c such that $p_U(n) = \mathrm{O}(n^c)$, then $a_U(n) \in \mathsf{RT}_1$.

6.2. **Labelled 0–1 Laws.** Compton shows that if ρ_L, the radius of convergence of $\mathbf{A}_L(x)$, is positive, then $\mathcal{K}$ has a labelled 0–1 law iff

$$\frac{a_L(n-k)/(n-k)!}{a_L(n)/n!} \to \infty \quad \text{whenever } p_L(k) > 0. \tag{10}$$

In particular it suffices to show that $a_L(n)/n! \in \mathsf{RT}_\infty$.

Compton's method to show that a given adequate class of finite relational structures $\mathcal{K}$ has a labelled 0–1 law is to show that its exponential generating function $\mathbf{A}_L(x) = \sum a_L(n)x^n/n!$ is Hayman-admissible with an infinite radius of convergence. This guarantees that $a_L(n)/n! \in \mathsf{RT}_\infty$ ([5], Corollary IV). However, as Compton notes, showing that $\mathbf{A}_L(x)$ is Hayman-admissible can be quite a challenge.

Question 8.3 of [3] first asks if, in the *unlabelled* case, the result of Bateman and Erdös, namely $p_U(n) \in \{0,1\}$ implies $a_U(n) \in \mathsf{RT}_1$, can be extended to the much more general statement that $p_U(n) = \mathrm{O}(n^k)$ implies $a_U(n) \in \mathsf{RT}_1$, yielding an unlabelled 0–1 law. As mentioned earlier, this was proved to be true by Bell. The second part of Question 8.3 asks if there is a simple sufficient condition along similar lines for the labelled case. We can now answer this in the affirmative with a result that is an excellent parallel to Bell's result for unlabelled structures.

Theorem 7. *If $\mathcal{K}$ is an adequate class of structures with*

$$p_L(n) \ =\ \mathrm{O}\Big(n^{\theta n}\Big) \quad \textit{for some } \theta \in (0,1)$$

[1]Given a logic $\mathcal{L}$, $\mathcal{K}$ has an unlabelled $\mathcal{L}$ 0–1 law means that for any $\mathcal{L}$ sentence φ, the probability that φ holds in $\mathcal{K}$ will be either 0 or 1. In [3] Compton worked with first-order logic, in [4] with monadic second-order logic. In both papers he simply used the phrases "unlabeled 0–1 law" and "labeled 0–1 law".

[2]This result is usually known as Schur's Theorem [6, 3.15.2]. One can easily find the asymptotics (9) using a partial fraction decomposition of the right side of (7). The labelled case with finitely many components is more difficult—we needed to invoke Hayman's treatise [5] just to obtain the asymptotics for $\log a_L(n)/n!$ (see Corollary 2).

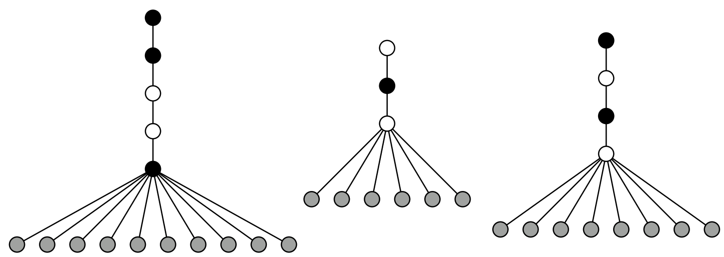

FIGURE 1. Brooms with two-colored handles

then $a_L(n)/n! \in \mathsf{RT}_\infty$, and consequently $\mathcal{K}$ has a labelled monadic second-order 0–1 law.

Proof. This is an immediate consequence of Theorem 5 and Compton's proof that $a_L(n)/n! \in \mathsf{RT}_\infty$ guarantees such a 0–1 law. □

Now we list the examples of classes $\mathcal{K}$ which Compton shows have a labelled 0–1 law, giving $p_L(n)$ in each case. It is trivial to check in each case that $p_L(n) = \mathrm{O}\Big(n^{n/2}\Big)$; thus the 0–1 law in each case follows from our Theorem 7.

(a) 7.1 Unary Predicates $p_L(n) = 0$ for $n > 1$.
(b) 7.12 Forests of Rooted Trees of Height 1 $p_L(n) = n$.
(c) 7.15 Only Finitely Many Components $p_L(n)$ is eventually 0.
(d) 7.16 Equivalence Relations $p_L(n) = 1$.
(e) 7.17 Partitions with a Selection Subset $p_L(n) = 2^n - 1$.

We can now augment this list by, in each case, coloring the members of $\mathcal{K}$ by a fixed set of r colors in all possible ways. This will increase the original $p_L(n)$ by a factor of at most r^n. This will still give $p_L(n) = \mathrm{O}\big(n^{n/2}\big)$. Furthermore, in each of these colored cases let $\mathcal{P}$ be any subset of the connected members, and let $\mathcal{K}$ be the closure of $\mathcal{P}$ under disjoint union. Each such $\mathcal{K}$ has a labelled 0–1 law.

Another application of Theorem 7 is to answer Question 4 of [3] by exhibiting an adequate class $\mathcal{K}$ such that $p_L(n) = \mathrm{O}\big(n^{3n/4}\big)$, hence there is a labelled 0–1 law for $\mathcal{K}$; but also such that $\rho_U \in (0,1)$, so $\mathcal{K}$ does not have an unlabelled 0–1 law.

Let the components of $\mathcal{K}$ be the one-element tree T_1 along with rooted trees T_{3n} of size $3n$ and height n consisting of a chain C_n of n nodes, with an antichain L_{2n} of $2n$ nodes (the leaves of the tree) below the least member of the chain; and the chain C_n is two-colored while the remaining nodes are uncolored. One can visualize these as brooms with 2-colored handles, see Figure 6.2.

The number of unlabelled components is given by $p_U(1) = 1$, $p_U(3n) = 2^n$. Thus the radius of convergence of the ordinary generating function of $\mathcal{K}$ is $\rho_U = \sqrt[3]{2}$. Since this is positive and not 1 it follows from Theorem 5.9(ii) of [3] that $\mathcal{K}$ does not have an unlabelled 0–1 law.

For the number $p_L(3n)$ of labelled components of size $3n$:

$$\begin{aligned} p_L(3n) &\le 2^n \binom{3n}{n} n! \\ &\le 2^n (3n)^n \exp\big(n \log n \cdot (1 + \mathrm{o}(1))\big) \\ &= \exp\big(2n \log n \cdot (1 + \mathrm{o}(1))\big) \\ &= (3n)^{(2/3)(3n)\big(1+\mathrm{o}(1)\big)} \\ &= \mathrm{O}\Big((3n)^{(3/4)(3n)}\Big). \end{aligned}$$

Thus $p_L(n) = \mathrm{O}\big(n^{3n/4}\big)$, so $a_L(n)/n! \in \mathsf{RT}_\infty$ by Theorem 7, showing that $\mathcal{K}$ has a labelled 0–1 law.

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